Extra questions for keen IA Nat. Sci. Maths Students

These questions are 'just for fun'. Some of them (or parts of them) I shall set as alternatives to questions from the lecturer's question sheet: in that case it is usually better to choose the question from the lecturer's sheet unless you really feel like a challenge. Some of them I shan't set at all. I shall mark any answers or attempts handed in, but I shan't spend very long talking about them in supervisions as they are somewhat tangential to the course. Questions marked with an asterisk are likelier to be more difficult than those without.

X. Assorted extension questions

X1. Take two identical cubes of side length a. Dissect the first into six congruent square-based pyramids, such that the base of each pyramid is obtained from a face of the cube, and the opposite vertex of each pyramid comes obtained from the centre. Attach one of these pyramids, by the base, to each of the faces of the second cube.

a) Show that the resultant solid has twelve faces, each a rhombus. It is called the *rhombic dodecahedron*.

b) Find the angles within the faces.

c) Find the angle between adjacent faces of the rhombic dodecahedron.

d) Show (using the result of the last part or using the original construction) that rhombic dodecahedra can tessalate to fill space.

e) (For materials scientists) Relate this tessalation to the cubic close-packed structure, and to the structure of diamond.

X2*. In crystallography, a triclinic unit cell is a parallelepiped forming the building block for certain kinds of crystal structures. We could describe it using the vectors **a**, **b**, and **c**, forming three of the edges starting from an origin at one corner. Crystallographers tend to us the lengths of the edges, $a = |\mathbf{a}|$, $b = |\mathbf{b}|$, and $c = |\mathbf{c}|$, along with the angles α, β, and γ, where α is the angle between **b** and **c**, β is the angle between **c** and **a**, and γ is the angle between **a** and **b**. Show that the volume of the unit cell is given by

$$V = abc\sqrt{1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma} \quad .$$

X3. Take a general quadrilateral $ABCD$, and construct squares on each side of it. Represent the points A, B, C, and D with the complex numbers a, b, c, and d. Represent the centres of the squares on AB, BC, CD, and DA with the complex numbers p, q, r, and s respectively.

a) Show that $p = a + \frac{1}{2}(b - a)(1 + i)$, where p is at the centre of the square on side AB.

b) Find similar expressions for q, r, and s.

c) Find $(p - r)/(q - s)$.

d) Hence show that the lines joining the centres of 'opposite' squares have equal length, and cross at right angles.

X4*. [Try X3 first!] Take a general triangle ABC, and construct equilateral triangles on each side of it. Let the equilateral triangle on side BC be $A'BC$. Define B' and C' similarly.

a) Represent points A, B, and C with the complex numbers a, b, and c. Work out the complex numbers corresponding to the centroids of the equilateral triangles $A'BC$, $B'AC$, and $C'AB$. Hence (or otherwise) show that the three centroids themselves form the vertices of a new equilateral triangle. [Hint: by what complex number must you multiply in order to keep the modulus the same, but rotate by 120°? Can you apply this to the sides of the new triangle?]

b) Join $A'A$, $B'B$, and $C'C$. Show (using complex numbers, or otherwise) that $A'A = B'B = C'C$, that the lines cut each other at 60°, and that they all meet at a point. This point is called the *Torricelli point* of the triangle ABC.

c) Draw a line through A perpendicular to $A'A$, though B perpendicular to $B'B$, and through C perpendicular to $C'C$. Continue these three lines to form a large triangle. Show that this large triangle is also equilateral.

d) Show that, in any equilateral triangle, the sum of the perpendiculars from a point to the three sides is a constant, independent of the point chosen.

e) By applying this result to the large triangle, show that, of all points P in triangle ABC, the point with the least total distance to the vertices, $PA + PB + PC$, is the Torricelli Point. (Finding which point in a triangle minimizes this sum was first set as a problem by Fermat.)

X5. A number is "the sum of two squares" if it can be written as $a^2 + b^2$, where a and b are integers. By considering $|(a+bi)(c+di)|^2$, show that, if p and q can each be written as the sum of two squares, their product, pq, can also be written as a sum of two squares.

X6. A parabola has the equation $y = x^2$. It is illuminated from above by rays of light parallel to the y-axis. These rays strike the parabola and are reflected. Find the gradients of the tangent and normal to the parabola at the point with x-co-ordinate x, and hence find the gradient of the reflected ray. Find the point f at which it crosses the y-axis, and show that this is independent of x. Comment on the shape of satellite dishes.

X7. A *cycloid* is the path traced by a point on a circle, if the circle rolls without slipping along a flat surface.

a) Find parametric equations describing the cycloid. [Hint: the easiest parameter to use is the angle through which the circle has turned.]

b) Find the length of one arch of the cycloid in terms of the diameter of the circle.

c) Find the area underneath one arch of the cycloid in terms of the area of the circle.

d) (*) An *intrinsic equation* is a relationship between s, the arc-length along a curve starting from the origin, and ψ, the angle between the tangent to the curve and the horizontal. (As such, $\tan\psi = \mathrm{d}y/\mathrm{d}x$; you should also be able to show that $\sin\psi = \mathrm{d}y/\mathrm{d}s$ and find a similar relation for $\cos\psi$.) Find the intrinsic equation of the cycloid.

X8. Derive a series expansion for

$$\log\left(\frac{1+x}{1-x}\right)$$

for small values of x. By choosing a suitable value of x, give a series for $\log 2$. Use this series to calculate $\log 2$ to four decimal places. (This series was used to calculate tables of logarithms in the days before computers were available to do the job.)

X9*. By considering the double integral

$$\int_{x=0}^{\infty}\int_{v=0}^{\lambda} e^{-ux}\cos vx\,\mathrm{d}v\,\mathrm{d}x$$

work out

$$\int_{x=0}^{\infty}\frac{\sin\lambda x}{x}\,\mathrm{d}x$$

and hence show that

$$\int_{-\infty}^{\infty}\frac{\sin x}{x}\,\mathrm{d}x = \pi \quad .$$

X10*. A car sets off due East at 20mph. The driver turns the steering wheel anti-clockwise at a constant rate, such that the front wheels of the car rotate three degrees (about vertical axes) per minute. Of course this cannot continue indefinitely, as the car will run out of steering-lock, but, if it could, the car would circle closer and closer in to one point.

a) Sketch the path of the car. The shape is known as *Cornu's Spiral*, and has applications in optics.

b) Show that the intrinsic equation of the spiral is of the form $\psi = k\pi s^2$, and find k, given that the wheelbase of the car (the distance between the front and back wheels) is 8ft 3in. (See the last part of question X7 for intrinsic equations.)

c) Give, in integral form, equations for the co-ordinates of the car's position in terms of the distance travelled, s.

d) By expressing the car's position as a complex number, find the position of the point towards which it tends as a Gaussian integral. Hence find how far away the point is from where the car starts, and in which compass direction it lies.

X11. A lighthouse is a distance b out to sea. It is due West of a cliff-top cottage on a certain coastline that runs exactly North-South. The lighthouse is broken, and randomly emits flashes of light in random directions (that is, the angle between the beam and the coastline is random). The beam is highly collimated (perhaps using a parabolic reflector as described in X6) so that each flash is essentially visible only at one point on the coast. The occupant of the cottage uses a long line of CCDs to determine where along the coastline each flash occurs, and records, each time, the distance from the cottage, x.

a) Work out the probability density function for x in terms of b. Make sure that it is normalized properly. Does it have a mean, median, mode, standard deviation, and interquartile range? Calculate each of these quantities that exists.

b) Work out (again for a given value of b) the probability density function that the first two flashes are received at x_1 and x_2. This is $P(x_1, x_2|b)$.

c) (*) Now look at the problem the other way round. Suppose that we've measured the first two flashes at positions x_1 and x_2. We want to use this information to work out an estimate for how far away the lighthouse is. Use Bayes' theorem. Start with a uniform prior (i.e. $P(b) = \text{const.}$, even though this cannot be normalized), as we start with no idea where the lighthouse is. Use the fact that $P(x_1, x_2)$ is independent of b to get an expression to which $P(b|x_1, x_2)$ is proportional, then normalize this to show that the probability density function for b is

$$P(b|x_1, x_2) = \frac{2(x_1 + x_2)b^2}{\pi(b^2 + x_1^2)(b^2 + x_2^2)} \quad .$$

[Hint: Use partial fractions to do the integration.]

d) (*) Does this new distribution have a mean, median, mode, standard deviation and interquartile range? Indicate how those quantities which exist would be calculated. Sketch the distribution, and work out the mode. How do you think that it is best to summarize the information that we have about the lighthouse's position?

e) (*) Suppose that more flashes are measured. What will this do to the probability distribution for b? Will it make any more quantities able to be calculated? (You needn't work out anything explicitly: the problem rapidly becomes rather complex.)

X12. A permutation is known as a *derangement* if it leaves none of the objects in their original places. (For example, there are $3! = 6$ permutations of three objects, because there are 6 ways of putting three cards, marked A, B, and C, in three envelopes, also marked A, B, and C. However, only 2 of them are derangements, because only 2 of them keep card A out of envelope A, card B out of envelope B, and card C out of envelope C.) Let $f(n)$ be the number of derangements of n objects: clearly, $f(n) < n!$.

a) Find $f(n)$ for $n = 1, 2, 3$, and 4.

b) By considering where the first object can go in a derangement, and what that leaves for the rest of the objects, show that $f(n) = (n - 1)[f(n-1) + f(n-2)]$. Hence work out $f(5)$ and $f(6)$.

c) (*) Using the last result and mathematical induction (or by a more cunning counting argument) show that $f(n) = nf(n-1) + (-1)^n$.

d) Using the previous result, show that

$$f(n) = n! \sum_{r=0}^{n} \frac{(-1)^r}{r!} \quad .$$

e) (*) Compare the series above with that for e^{-1}, and hence show that $f(n)$ is always the integer closest to n/e.

f) A large number n of Christmas cards is to be sent to n recipients. A card and envelope are written for each recipient. Unfortunately, due to frost damage to the writer's glasses, the cards are put into the envelopes entirely at random. Give the approximate probability that no-one gets the right Christmas card, and the order of the error. Also work out the (exact) mean and standard deviation of the number of correctly-delivered cards. Compare this to a well-known standard distribution, and comment.

X13. In the Solar system, a small object of mass m, far enough away from the planets to be unaffected by their gravitational field, is released. Initially it is at rest a distance x_0 from the Sun. If M is the mass of the Sun, G the gravitational constant, x the distance of the object from the Sun, and t the time since release, Newton's laws give

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{-GM}{x^2} \quad .$$

Solve this differential equation to find how long the object takes to reach a certain distance x from the Sun. *[Hint: try re-casting the equation in terms of the velocity, v, and x.]* Hence show that the time taken to reach the centre of the Sun is

$$\pi \sqrt{\frac{x_0^3}{8GM}} \quad .$$

X14. Differential equations with constant coefficients are susceptible to similar methods as the order of the the differential equation increases.

a) Find the general solution of

$$\frac{\mathrm{d}^4 y}{\mathrm{d}x^4} = ky$$

treating separately the cases $k > 0$, $k = 0$, and $k < 0$. (Fourth-order differential equations describe the deflexion of beams, amongst other things. This one would apply to a uniformly tapering beam.)

b) Find the general solution of

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y = \sin^2 x \quad .$$

X15. Find the stationary points of the function

$$z = (3xy^2 - x^3)e^{-\frac{3}{2}(x^2+y^2)}$$

and the values of z at those points. Classify the stationary points and provide a contour sketch of the function for x and y between -2 and 2 or so.

X16. Recall the *cyclic relation* between the partial derivatives of three variables, which applies when they are related together by one constraint so that there are two degrees of freedom. Try to view the cyclic relation geometrically if possible.

a) When there are four variables related by only one constraint, we can form partial derivatives like $\left.\frac{\partial w}{\partial x}\right|_{y,z}$. Find the value of

$$\left.\frac{\partial w}{\partial x}\right|_{y,z} \left.\frac{\partial x}{\partial y}\right|_{z,w} \left.\frac{\partial y}{\partial z}\right|_{w,x} \left.\frac{\partial z}{\partial w}\right|_{x,y} \quad .$$

b) (*) When there are four variables related by two constraints the cyclic form doesn't have a fixed numerical value. Find, nonetheless, a formula for

$$\left.\frac{\partial w}{\partial x}\right|_{y} \left.\frac{\partial x}{\partial y}\right|_{z} \left.\frac{\partial y}{\partial z}\right|_{w} \left.\frac{\partial z}{\partial w}\right|_{x}$$

in terms of only $\left.\frac{\partial w}{\partial x}\right|_{y}$ and $\left.\frac{\partial w}{\partial x}\right|_{z}$.

X17.

a) The scalar fields ρ and p and the vector field $\mathbf{F}$ are related by $\rho\mathbf{F} = \boldsymbol{\nabla} p$. Show that $\mathbf{F}$ is perpendicular to its own curl.

b) The vector field $\mathbf{F}$ is solonoidal, i.e. $\boldsymbol{\nabla}\cdot\mathbf{F}=0$. The operator ∇^2 can be defined when acting on vectors by applying the usual $\nabla^2 \equiv \operatorname{div}\operatorname{grad} \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ operation to each component to make a new vector. Show that

$$\operatorname{curl}\operatorname{curl}\operatorname{curl}\operatorname{curl}\mathbf{F}=\nabla^4\mathbf{F} \quad .$$

c) From the divergence theorem derive Green's theorem ("Green's second identity"),

$$\oint_{\partial V}(u\boldsymbol{\nabla}v - v\boldsymbol{\nabla}u)\cdot\,\mathrm{d}\mathbf{S}=\int_V u\nabla^2 v - v\nabla^2 u\,\mathrm{d}\tau$$

where ∂V denotes the closed surface bounding a region of space V, and $\mathrm{d}\tau$ is an element of volume. [Note that the standard notation amongst mathematicians is to make the number of integral signs match the number of d's, though in applied work we sometimes write more integral signs to remind ourselves of the dimensionality of the integration.]

X18. The water in an emptying bathtub is spiralling around the plug-hole. Consider the velocity of the water at the surface (assumed flat) to be a vector $\mathbf{v}$ within the x-y plane. Let the plug-hole be at the origin. The laws of fluid dynamics for a low viscosity fluid such as bathwater imply that the *vorticity* of the fluid, defined as the curl of the velocity (i.e. $\boldsymbol{\nabla}\times\mathbf{v}$) is negligible unless the point under consideration is very close to the edges of the bath or to the plug-hole.

Assume that the flow is purely tangential. (This can't be quite true, or the bath would never empty, but it's not a bad approximation.) By considering the effect of Stokes' Theorem on the velocity field, find expressions for the two components of $\mathbf{v}$ in terms of the position co-ordinates x and y and a single unknown parameter which represents the strength of the vortex. This expression will be valid close, but not too close, to the plug-hole.

X19. Consider a Fourier series of period 2π where the constant term is $\frac{1}{2}$, all of the coefficients of the cosine terms are 1, and all the coefficients of the sin terms are 0. (In this question, unusually, we are going to start by knowing the coefficients, and seek to understand the underlying function.)

a) First truncate the series after the $\cos(2x)$ term. *Include only half of the* $\cos(2x)$ *term itself.* Sketch this function between 0 and 4π.

b) Now include one more term, truncating the series after $\cos(3x)$. (Include only half of this last term.) Sketch this function over the same range. If you have a computer to hand, look at similar plots with more terms.

c) Now try to understand the behaviour of the series algebraically. Find an expression for the sum of the series up to the $\cos(Nx)$ term, including only half of the $\cos(Nx)$ term itself. Show that the expression reduces to $\frac{1}{2}\cot(\frac{x}{2})\sin(Nx)$. Explain what happens to this number (i) when x is not close to a multiple of 2π and (ii) when x is close to a multiple of 2π.

d) Now try to understand the behaviour of the series geometrically. Again including only half of the final term before truncating the series, show the numbers $\frac{1}{2}$, e^{ix}, e^{2ix} &c. on an Argand diagram for a small value of x and a random larger value of x. Arrange the vectors nose-to-tail to form the complex sum, and relate the real part of this complex number to the behaviour observed in the previous part. (For physicists, relate also to multi-slit diffraction, and for materials scientists, relate to Bragg reflexion.)

e) What must the integral of the Fourier series be over a complete period? (Only one term in the series contributes.)

f) What will the integral be of the Fourier series over any range that does not include a multiple of 2π?

g) (*) Discuss the form of the function itself, and explain how the coefficients would have been obtained from this function.

X20. This question is about Vandermonde determinants, which have applications in polynomial interpolation.

a) Show that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

is equal to $(y-x)(z-x)(z-y)$.

b) Find

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ w & x & y & z \\ w^2 & x^2 & y^2 & z^2 \\ w^3 & x^3 & y^3 & z^3 \end{vmatrix}$$

in a similar form.

c) (*) Generalize this result to n variables, raised to powers $0, 1 \ldots n-1$ in an n by n determinant.

X21. Reflexions of objects in mirrors appear to have their left and right halves swapped round. Why, given the isotropy of space, is this the case when the top and bottom halves aren't swapped?