IA NST Post-Christmas College Maths Test

January 2019

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new piece of paper, writing your name and the question number clearly at the top. Do not submit answers to more than two questions. Calculators may not be used for this test.

Question 1

Find the *real part* of the principal value of each of the following complex numbers. Give each answer in explicitly real form.
  a) i^7 [1]
  b) $\sqrt[7]{i}$ [3]
  c) 7^i [3]
  d) $\sin(7+i)$ [3]
  e) $^{i-7}\sqrt{i-1}$ [6]
  f) $\log_e(\sin(i-7))$ [4]

Question 2*

$$\text{Let} \quad f(x,y) = \frac{|x||y|}{1+x^2+y^2} \quad .$$

  a) Find the integral of f over the circle of radius R, centred on the origin, in the x-y plane. [6]
  b) Find the integral of f over the square in the x-y plane defined by $-R \le x \le R$ and $-R \le y \le R$. [9]
  c) Find the limit of the ratio of your answers to (a) and (b) as R tends to infinity. [5]

Question 3

The Museum of Probability contains a three-dimensional Cartesian co-ordinate system. It is defined by the position and orientation of three brass rods which are held on a pole in one of the rooms of the Museum. The rods form the axes and their intersection is the origin. The rods are fixed securely to each other at right angles, but their attachment to the supporting pole is less secure. Initially, the **i**-, **j**-, and **k**-axes point due East, due North, and vertically upwards respectively.

One day, the Museum's goat knocks into the pole on which the axes are supported, causing them to point in new directions. (The Museum's goat is used in practical demonstrations of the Monty Hall problem. However, it often causes trouble.) In the rest of this question, all co-ordinates are given relative to the axes after they have been knocked into by the goat.

a) The three points $(-1, 4, 0)$, $(-2, 0, 0)$, and $(0, 1, -1)$ are found to lie on the (perfectly horizontal) floor of the Museum. Find, in both Cartesian and vector form, an equation for the floor. [5]

b) Find the height of the origin above the floor. [2]

c) The two points $(-1, 1, 1)$ and $(0, 2, -1)$ are found to lie on the inner surface of a wall of the Museum. This wall is perfectly vertical and runs perfectly aligned in an East-West direction; it can be reached by walking North from the origin. Find, in both Cartesian and vector form, an equation for the surface of this wall. [5]

d) Find a unit vector that points due East horizontally. [2]

e) The Director decides to attach the origin of co-ordinates to the line joining the floor described in (a) to the surface of the wall described in (c). Find the minimum length of string needed. [6]

Question 4

Every year on 17th December from 7am to 10am, N students sit examination E, in which calculators are forbidden. Nevertheless, every year C of them bring calculators with them. Each year the Invigilator checks a random sample of n students. Whenever the Invigilator checks a student, he always finds any calculator that the student has. He punishes any student who has a calculator, but does not take away the calculator. All of the checks (and punishments) take place before the examination starts.

a) In 2015, the Invigilator carefully makes a random selection of n distinct students to check. Find the probability that he finds exactly r calculators. [4]

b) In 2016 and all subsequent years, the Invigilator decides that it is too much trouble to remember which students he has checked. Therefore he selects the n students one at a time, randomly from all of the N students such that each selection is independent of the others. This means that the same student may be checked more than once; a student with a calculator may be punished more than once for the same offence. Find the probability that he finds calculators on precisely r of the n checks. [3]

c) Because media reports suggest that the nature of the punishment is too extreme, the Invigilator is required from 2017 onwards to fill in a form (which takes him a time t_F) for each punishment that he administers. Find the mean and standard deviation of the time that he spends filling in these forms. [3]

d) University officials notice that the mean in (c) increases with increasing n, and that this creates an incentive for the Invigilator to use a smaller value of n, which risks creating an inadequate deterrent. Beginning in 2018, they therefore appoint a large number of Proctors, who inspect examination rooms while the examinations are underway, arriving randomly and independently of each other, such that the probability that at least one proctor arrives during examination E is p. Find the mean and standard deviation of the number of Proctors arriving during the 2018 examination. [4]

e) If at least one Proctor arrives during the examination, and the Invigilator has failed to find any calculators at all, then the Invigilator is required to attend a boring course, which takes time t_{BC}. Find the expected value of the time spent by the Invigilator on the course in 2018. [2]

f) The Invigilator wishes to minimize the sum of the expected values of time spent on forms and courses, that is, the sum of the means from (c) and (e). Find the optimum value of n. [4]